NPS-55MG70101A

United States Naval Postgraduate School

APPRAISING FEASIBILITY AND MAXIMAL FLOW CAPACITY OF A NETWORK

by

Alan W. McMasters

23 October 1970

NAVAL POSTGRADUATE SCHOOL
Monterey, California

Rear Admiral R. W. McNitt, USN
Superintendent

M. U. Clauser
Academic Dean

ABSTRACT:

The use of the dual graph in determining the value of the maximal flow capacity of an undirected network has been extended to directed networks. A directed dual graph is defined such that the length of the shortest route through this dual is equal to the maximal flow capacity of its directed primal. Feasibility of a specified exogenous flow for networks having positive lower bounds on arc flows can also be appraised. Infeasibility is indicated by a dual cycle of negative length.

NPS-55MG70101A

23 October 1970

TABLE OF CONTENTS

1. INTRODUCTION

Ford and Fulkerson [2] have suggested that an easy way to find the value of the maximum possible flow through an undirected network is to construct the dual graph, assign the capacities of the intersected primal arcs as lengths of arcs in the dual, and then find the shortest route through the dual. The length of this shortest route represents the value of the undirected minimum cut set. From the min-cut max-flow theorem we then know this length is equal to the value of the maximum possible flow through a network. The purpose of this paper is to extend this idea to flows in directed networks; particularly networks having nonzero positive lower bounds on arc flow.

2. MAXIMAL FLOW IN AN UNDIRECTED NETWORK

We begin with a statement of a procedure for constructing the dual graph of an undirected network [4].

1. Denote the original maximal flow network as the primal network. Connect an artificial arc between the sink and source of the primal. The network will then be referred to as the modified primal network.

2. Place a node in each mesh of the modified primal including the external mesh. Let the origin of the dual be the node in the mesh involving the artificial arc and the destination be the node in the external mesh.

3. For each arc in the primal (except the artificial arc) construct an arc that intersects it and joins with nodes in the meshes adjacent to it.

4. Assign each arc of the dual a length equal to the capacity of the primal arc it intersects.

The shortest route through the dual can be determined by first assuming every arc of the dual can be replaced by two oppositely directed arcs of the same length and then applying any one of the well known shortest route algorithms. Several are given by Dreyfus [1].

By way of example, consider the undirected network shown in Figure 1. The number beside each arc is its undirected flow capacity M_{ij}. Any flow X_{ij} through an arc (i,j) is therefore bounded according to (1).

$$-M_{ij} \le X_{ij} \le M_{ij} \qquad (1)$$

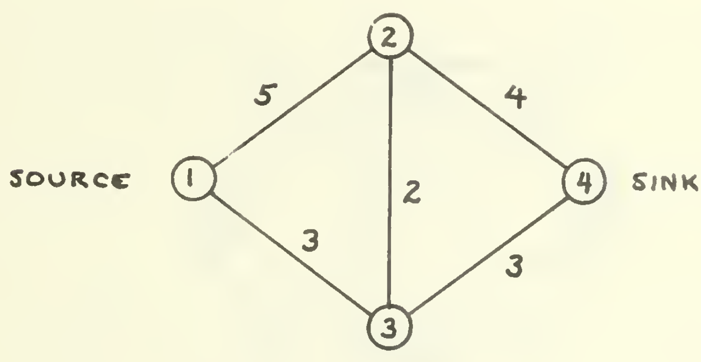

Figure 1.

The construction of the dual is shown in Figure 2 where arc (4,1) is the artificial arc added to facilitate the construction process. The undirected dual consists of the nodes A, B, C, and D and the dashed arcs shown in Figure 2. Node A will be the origin and node D will be the destination.

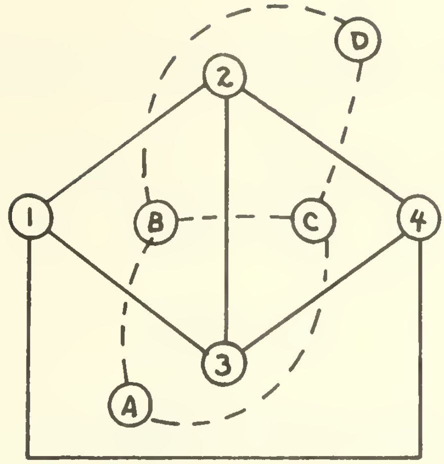

Figure 2.

The shortest route problem is then to find the shortest route from node A to node D if the undirected arcs of the dual are replaced by oppositely directed arcs having lengths corresponding to flow capacities of the primal arcs they intersect. Figure 3 shows the dual of the example in directed form ready for determining the shortest route.

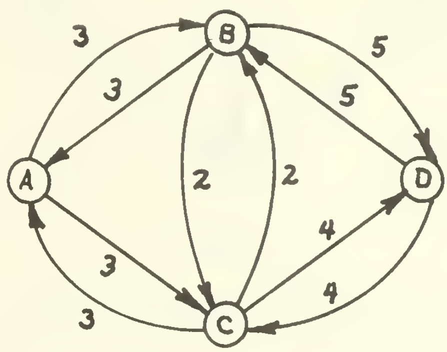

Figure 3.

The shortest route from A to D is A-C-D with a length of 7. Therefore the maximum possible flow through the network of Figure 1 is 7 units. Because the arcs of the primal intersected by the arcs from the shortest route of the dual are the minimum cut set of the primal [2], we know that arcs (2,4) and (3,4) are the minimum cut set for this example.

3. MAXIMAL FLOW IN A DIRECTED NETWORK

If the network of Figure 1 is changed to the directed form shown in Figure 4 where arc flow is restricted to

$$0 \le X_{ij} \le M_{ij}, \tag{1}$$

then the structuring of the dual shortest route problem requires some care. The construction of the dual is the same in the initial phases as that shown in Figure 1. However, there cannot be two oppositely directed arcs of the same length if the lengths of the routes through the dual are to correspond to values of the cut sets of the primal.

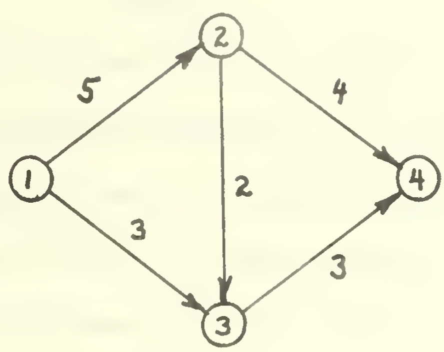

Figure 4.

The following convention will be used for determining the lengths of the dual arcs. Beginning with the arcs incident with the dual source, assign the length M_{ij} to that dual arc having the same direction as if the crossed primal arc had been rotated 90 degrees counterclockwise. Then assign a length of zero to the dual arc oppositely directed. Thus, the arc (A,B) in Figure 3 should have a length of 3 and the arc (B,A) should have a length of zero because arc (1,3) is directed from node

1 to node 3. The arc (B,C) should have a length of 2 and the arc (C,B) should have a length of zero because arc (2,3) is directed from node 2 to node 3. This convention results in the dual shortest route network shown in Figure 5.

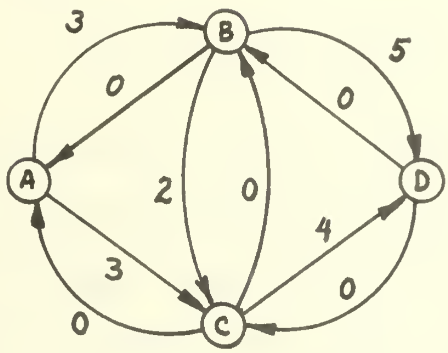

Figure 5.

If the reader reflects for a moment on what we have just done and compares it with the case of maximum flow in undirected arcs he will realize that we have changed one arc of each pair of dual arcs from a length of M_{ij} to that of zero. An undirected arc (i,j) can be thought of as having a flow capacity restriction of the form given by (2),

$$-M_{ij} \leq X_{ij} \leq M_{ij}, \tag{2}$$

while the flow restrictions for arcs in a digraph are usually of the form given by (1). The change we have made has been only the length of the arc corresponding to the lower bound.

We immediately realize that the arc corresponding to the lower bound could have any value of length. For example, there is nothing to restrict us to having a lower bound on primal arc flow which is the

negative of M_{ij} when it could be any other negative number (even minus infinity). The dual arc would then have a positive length equal to the negative of the lower bound value.

This fact suggests a way of looking at flow problems having positive lower bounds; that is,

$$0 < L_{ij} \leq X_{ij} \leq M_{ij}. \tag{3}$$

Following our observations above we would assign the "lower bound" arc of the dual pair intersecting primal arc (i,j) a length value of $-L_{ij}$. This results in one or more routes through the dual having lengths consisting of a sum of positive M_{ij} values and negative L_{ij} values.

Now the value of a generalized cut set when lower bounds are positive is given by equation (4) where the set S contains the source node; the set $\bar{S}$ contains the sink node [3]. Thus, our convention of assigning the negative of the L_{ij} values to the dual "lower bound" arcs provides any route without cycles through the dual with a length corresponding to the value of the generalized cut set intersected by those dual arcs.

$$V(S,\bar{S}) = \sum_{\substack{i\varepsilon S \\ j\varepsilon \bar{S}}} M_{ij} - \sum_{\substack{i\varepsilon S \\ j\varepsilon \bar{S}}} L_{ij} \tag{4}$$

By way of example, suppose the graph of Figure 4 has positive lower bounds added to it as shown in Figure 6. The numbers on each arc are L_{ij}, M_{ij}.

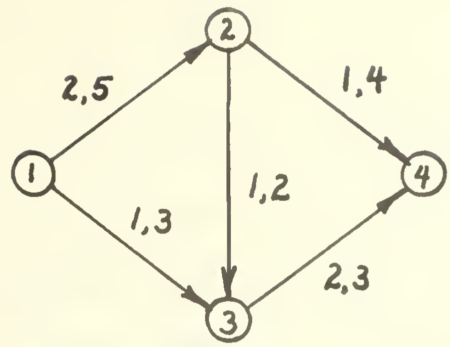

Figure 6.

The dual of Figure 6 is shown in Figure 7.

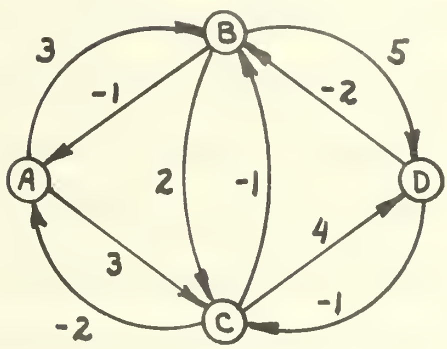

Figure 7.

From Figure 6 we know that the value of the cut set consisting of arcs (1,2), (2,3), and (3,4) is

$$M_{12} + M_{34} - L_{23} = 5 + 3 - 1 = 7. \tag{5}$$

The route through the dual network which corresponds to this cut set is the chain [(A,C),(C,B),(B,D)] which has a length given exactly by equation (5).

A minimum route in Figure 7 happens to be the chain [(A,C),(C,B),(B,D)] so that the maximum possible flow through the primal network is, in fact, 7 units.

4. THE QUESTION OF FEASIBILITY

For problems having at least one arc with a positive lower bound the question of feasibility is of importance. We consider two forms of the question. If we ask, "What is the maximal possible flow through a network?", then we can analyze the shortest route through the dual to see if it is nonnegative or not. Its length, if positive, will be the value of the maximum possible flow; if zero, a feasible flow within the network is possible but no exogenous flow can leave or enter the network. If the problem has no feasible flow then any shortest route algorithm used will detect a cycle of negative length implying that the shortest route has an infinite negative length. Infeasibility in the primal maximal flow problem means that some lower bound is forcing a flow which would exceed an upper bound somewhere else in the network.

An example of an infeasible maximal flow network is shown in Figure 8. The arcs (2,3) and (2,4) are lower bounded with values of 1 and 4, respectively. Only arc (1,2) can provide flow to these arcs. Unfortunately, (1,2) has an upper bound value of 2 which prevents the needed 5 units of flow from reaching (2,3) and (2,4).

The dual for the problem is shown in Figure 9. One cycle of negative length consists of arcs (B,D), (D,C), and (C,B) with a length of -3 which is the difference between the maximum amount that

arc (1,2) can provide and the minimum amount needed by arcs (2,3) and (2,4).

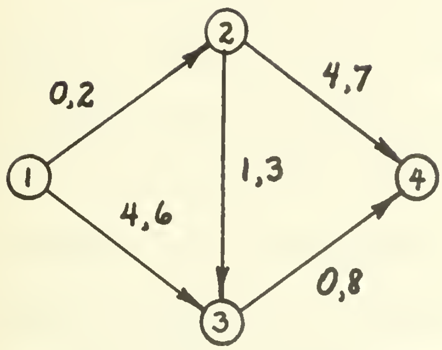

Figure 8.

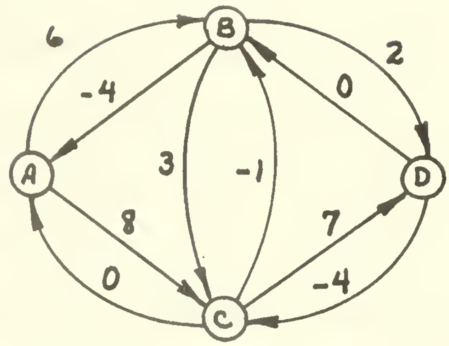

Figure 9.

More complicated examples involving less obvious infeasibilities can be developed. The network, of Figure 10 for example, is infeasible because the sums of the lower bounds on arcs (1,2) and (1,3) exceed the upper bound on arc (4,5). The dual is shown in Figure 11. A cycle of negative length consists of (A,C), the upper (C,B) arc, and either the upper or lower (B,A) arcs. The length of this cycle is -1 which we realize is the amount of capacity that arc (4,5) is short.

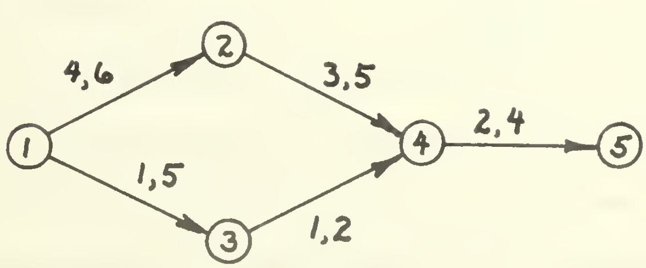

Figure 10.

Let us now turn to the second form of the feasibility question; "Is it possible to send a flow of a specified amount through the network?" If a network has all zero lower bounds on arc capacities then knowing maximal flow we can also answer this question. If the specified flow is larger than the maximal capacity the problem is infeasible; if it is less than or equal to the maximum capacity then the problem is feasible. If a network has positive lower bounds on arc flows then knowing the value of the maximal possible flow allows us to answer the question only if the specified flow is greater than or equal to the maximal possible. If the specified flow is less then there is no guarantee that it will be feasible.

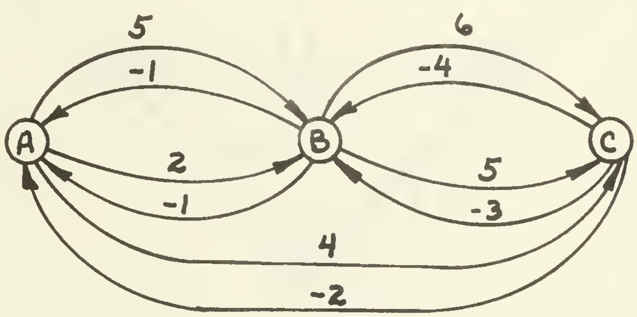

Figure 11.

Consider, for example, the network in Figure 12. The maximal possible flow which can be sent from node 1 to node 2 is 4; the minimal possible feasible flow is 1. An internal circulation of 2 units meets the flow feasibility requirement of arc (2,3) and arc (3,1) as well for the maximal flow problem. For the minimal flow problem an internal circulation of 3 units is required. These three units plus the one exogenous unit meet the flow feasibility requirement for arc (1,2). Thus, a specified Q is feasible only if it is in the interval $1 \leq Q \leq 4$. Otherwise it will be infeasible.

The feasibility question involving specified Q can be easily handled using the dual shortest route approach if we add an arc directed from the sink to the source having lower and upper bounds on its flow equal to the specified Q. If this specified Q is not feasible then a cycle of negative length will appear. If the specified Q is maximal feasible, then the length of the shortest route through the dual will be equal to Q. If the specified Q is feasible but the maximal possible flow is larger then the length of the shortest route through the dual will be the value of the maximal flow. To illustrate these features, we add the arc (2,1) to Figure 12 and Figure 13.

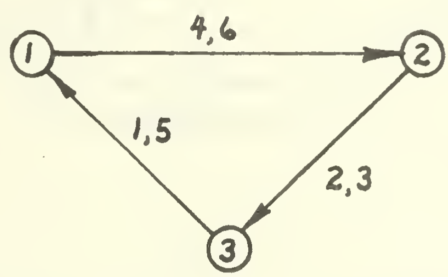

Figure 12.

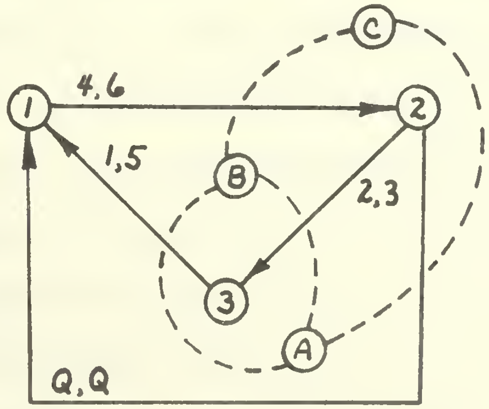

Figure 13.

The addition of an artificial arc is not necessary for the construction of the dual when an arc already exists from the sink to the source. We merely designate the dual origin as the node in the mesh associated with the primal sink to source arc. It should be noted, however, that we must have a dual arc to intersect the legitimate sink to source arc. The initial phase of the dual construction is shown in Figure 13. The final dual form is shown in Figure 14.

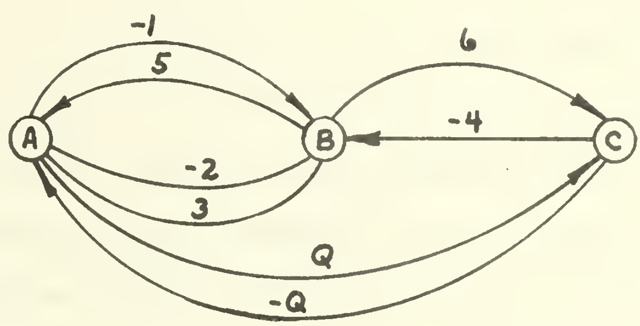

Figure 14.

Suppose now that the value of Q is larger than 4. The only cycle of negative length from A back to A consists of the lowest (A,B) arc, arc (B,C), and arc (C,A). The length of this cycle is 4 - Q. As Q decreases to 4 this cycle length goes to zero. For all Q less than 4 its length remains positive.

Suppose next that $Q < 1$. The only cycle of negative length from A back to A consists of the arc (A,C), arc (C,B), and the lower (B,A) arc. The length of this cycle is Q - 1. As Q increases to 1 this cycle length goes to zero. For all $Q > 1$ it has positive length.

Ford and Fulkerson [3] present a circulation theorem (theorem 3.1) which specifies that a network flow problem will be feasible if and only if

$$\sum_{\substack{i \varepsilon Y \\ j \varepsilon \bar{Y}}} M_{ij} \geq \sum_{\substack{i \varepsilon \bar{Y} \\ j \varepsilon Y}} L_{ij} \qquad (6)$$

for all cut sets $(Y,\bar{Y})$ of a network which has an arc $(n,1)$ with lower and upper bounds on its flow. The addition of $(n,1)$ creates a sourceless-sinkless network.

If we examine the dual as shown imposed in Figure 13 we see that every circuit corresponds to a cut set of the type Ford and Fulkerson address in their circulation theorem. The inequality (6) is nothing more than another way of writing (4) for a sourceless-sinkless network. The negative cycles we encountered in Figure 14 are due to the generalized cut set value in question being negative for the specified Q value. Therefore, a negative cycle indicates a cut set which violates (6). And, as the theorem states, only one such cut set is needed to make a problem infeasible.

Finally, the sourceless-sinkless structure can also be used to answer the question of whether any nonnegative flow would be feasible. In that case we would assign zero as the lower bound and infinity as an upper bound on the arc from the sink to the source. The value of the maximal possible feasible flow would still appear as the length of the shortest route from the origin to the destination of the dual. Infeasible networks would cause a cycle of negative length in the dual.

REFERENCES

[1] Dreyfus, S. E., "An Appraisal of Some Shortest Path Algorithms," *Operations Research*, Vol. 17, No. 3, May-June 1969.

[2] Ford, L. R. and D. R. Fulkerson, "Maximal Flow Through a Network," *Canad. Jour. Math.*, Vol. 8, No. 3, 1956.

[3] Ford, L. R. and D. R. Fulkerson, *Flows in Networks*, Princeton University Press, Princeton, New Jersey, 1962.

[4] Wollmer, R. D., "Removing Arcs from a Network," *Operations Research*, Vol. 12, No. 6, November-December 1964.

INITIAL DISTRIBUTION LIST

Defense Documentation Center (DDC) Cameron Station Alexandria, Virginia 22314 Attn: IRS	20
Library Naval Postgraduate School Monterey, California 93940	2
Library Department of Operations Analysis Naval Postgraduate School Monterey, California 93940	1
Professor Alan W. McMasters Code 55Mg Naval Postgraduate School Monterey, California 93940	50
Dean of Research Administration Code 023 Naval Postgraduate School Monterey, California 93940	2
Central Files Naval Postgraduate School Monterey, California 93940	2

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Naval Postgraduate School Monterey, California	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE

Appraising Feasibility and Maximal Flow Capacity of a Network

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Technical Report 1970

5. AUTHOR(S) (First name, middle initial, last name)

McMasters, Alan W.

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
23 October 1970	20	4

8a. CONTRACT OR GRANT NO.	9a. ORIGINATOR'S REPORT NUMBER(S) NPS-55MG70101A
b. PROJECT NO.	
c.	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)
d.	

10. DISTRIBUTION STATEMENT

This document has been approved for public release and sale; its distribution is unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY

13. ABSTRACT

The use of the dual graph in determining the value of the maximal flow capacity of an undirected network has been extended to directed networks. A directed dual graph is defined such that the length of the shortest route through this dual is equal to the maximal flow capacity of its directed primal. Feasibility of a specified exogenous flow for networks having positive lower bounds on arc flows can also be appraised. Infeasibility is indicated by a dual cycle of negative length.

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Network flows						
Graph theory						
Feasibility						
Optimization						